Radiative corrections in weak semi-leptonic processes at low energy: a two-step matching determination

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Radiative corrections in weak semi-leptonic processes at low energy: a two-step matching determination [1]

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Abstract

We focus on the chiral Lagrangian couplings describing radiative corrections to weak semi-leptonic decays and relate them to the decay amplitude of a lepton, computed by Braaten and Li at one loop in the Standard Model. For this purpose, we follow a two-step procedure. A first matching, from the Standard Model to Fermi theory, yields a relevant set of counterterms. The latter are related to chiral couplings thanks to a second matching, from Fermi theory to the chiral Lagrangian, which is performed using the spurion method. We show that the chiral couplings of physical relevance obey integral representations in a closed form, expressed in terms of QCD chiral correlators and vertex functions. We deduce exact relations among the couplings, as well as numerical estimates which go beyond the usual $\log(M_Z/M_\rho)$ approximation.

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1 Introduction

An accurate evaluation of radiative corrections in K_{l3} decays is crucial for a precise determination of V_{us}. In this context, it is necessary to control whether experimental data on K_{l3}^+ and K_{l3}^0 data are consistent [1]. Several new experiments have studied K decays. Results on the K_{l3}^+ mode were released very recently by the E865 [2] and ISTRA [3] collaborations and results on the K_{l3}^0 mode were presented by the NA48 [4], KTev [5] and KLOE [6] collaborations. This has stimulated renewed interest in the theoretical determination of radiative corrections in such processes [7, 8].

This subject has a long history [9]. Within the Standard Model, a conspicuous feature of radiative corrections to semi-leptonic decays is their enhancement by a large logarithm $\log(M_Z/\mu)$ with $\mu \simeq 1$ GeV, which was pointed out by Sirlin [10, 11]. In this paper, we will focus on the remaining (unenhanced) corrections and will discuss a method for determining them. The proper theoretical framework to discuss semi-leptonic decays of kaons (as well as those of π's or η's) is the chiral effective Lagrangian formalism [12, 13, 14] (see the book [15] for a review of applications). The discussion of radiative corrections requires extensions of the original setting which were performed successively by Urech [16] and then by Knecht *et al.* [17]. At this stage, the effective Lagrangian includes not only the pseudo-Goldstone bosons, but also the photon and the light leptons as dynamical degrees of freedom. In other words, this Lagrangian describes the whole Standard Model at low energies. High-energy dynamics has been integrated out into local (contact) terms, parameterised by a set of low-energy constants (LEC's).

In this paper, we will consider the set of LEC's X_i introduced in ref. [17] to deal with virtual leptons and discuss their physical interpretation. In particular, we will show that they satisfy simple integral representations in terms of QCD Green functions in the chiral limit. These results extend those obtained in the case of the Urech LEC's K_i for virtual photons [18], which were themselves generalisations of the well-known sum rule by Das *et al.* [19]. These integral representations provide practical means of estimating the LEC's X_i numerically, once the chiral Green functions are approximated by simple, large-N_c motivated, models. But our analysis goes beyond these numerical results, since we will derive some non-trivial relations among the LEC's X_i and with the electromagnetic coupling K_{12}. This will allow us to clarify completely a related issue, the dependence of K_{12} on short-distance renormalisation conditions, observed in [18] and further discussed in [20]. We start from a result of Braaten and Li (denoted BL in the following) [21], who computed the amplitude for a lepton[4] decaying into a massless quark, a massless antiquark, and a neutrino at one loop in the Standard Model. This computation completed earlier results obtained by Sirlin [11].

We will follow a two-step matching procedure which can be sketched as:

$$\boxed{\text{Standard Model}} \longrightarrow \boxed{\text{Fermi theory}} \longrightarrow \boxed{\text{Effective Lagrangian}}$$

[4]The authors of ref. [21] were chiefly interested in the case of the τ lepton. However, their result is general and it will be applied to the light leptons e and μ here.

This two-step procedure will allow us to determine the implications of BL's calculation for the effective Lagrangian. It turns out to be particularly convenient to introduce Fermi theory as an intermediate stage, in order to integrate out the high-energy dynamics of the Standard Model in a transparent way. In addition, at this intermediate stage, we can rely on a Pauli-Villars regularisation (applied to the photon propagator) to tame divergences. This regularisation scheme offers the attractive feature of remaining in four dimensions, and thus avoids the well-known difficulties of dimensional regularisation when defining γ_5. This will prove particularly useful when we deal with chiral QCD correlators.

The plan of the paper is as follows. We begin by reconsidering the one-loop calculation of radiative corrections to the semi-leptonic decay of a light lepton in Fermi theory. The ultraviolet divergences are removed through a set of counterterms. Matching the one-loop amplitude in the Standard Model and in Fermi theory yields constraints on the values of the latter. Then, we re-express the counterterms in Fermi theory to introduce spurion sources instead of the electric and weak charge matrices. Using this new form, we perform the second matching step and identify counterterms in Fermi theory and LEC's in the chiral Lagrangian. This identification involves also chiral two- and three-point Green functions. Finally, integral relations are derived, which are exploited to obtain analytical relations among chiral LEC's and numerical estimates based on large-N_c models for the relevant chiral correlators.

2 One-loop matching of Fermi theory and the Standard Model revisited

2.1 Tree-level amplitude

Following ref. [21], we consider the amplitude $T(p,q,p',q')$ for the semi-leptonic weak decay of a lepton into massless quark, antiquark and neutrino

$$l(p) \to \bar{u}(q) + d(q') + \nu(p') \ . \tag{1}$$

The usual kinematical variables are introduced

$$s = (p-q)^2, \quad t = (p-p')^2, \quad u = (p-q')^2, \quad s+t+u = M_l^2 \ . \tag{2}$$

BL have computed the one-loop amplitude $T(p,q,p',q')$ in the Standard Model, and we intend to perform the same work in Fermi theory in the presence of electromagnetic interactions (see fig. 1). The relevant part of the interaction Lagrangian is

$$\mathcal{L}_{Fermi} = -\frac{4G_F V_{ud}}{\sqrt{2}} \left\{ \bar{l}_L \gamma^\lambda \nu_L \times \overline{d}_L \gamma_\lambda u_L + h.c. \right\} \ , \tag{3}$$

At leading order, we consider the diagram in fig. 1, which gives the following results for the amplitude

$$T_0 = -\frac{G_F V_{ud}}{\sqrt{2}} \bar{u}_\nu(p') \gamma_\lambda (1-\gamma^5) u_l(p) \, \bar{u}_d(q') \gamma_\lambda (1-\gamma^5) v_u(q) \ , \tag{4}$$

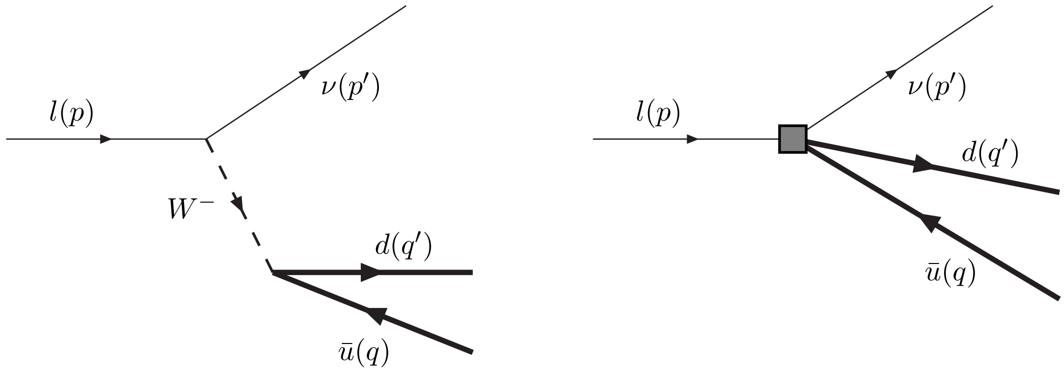


Figure 1: Tree-level diagram for the semi-leptonic decay of a lepton into a massless quark, antiquark and neutrino in the Standard Model (left) and in Fermi theory (right).

and for the decay rate

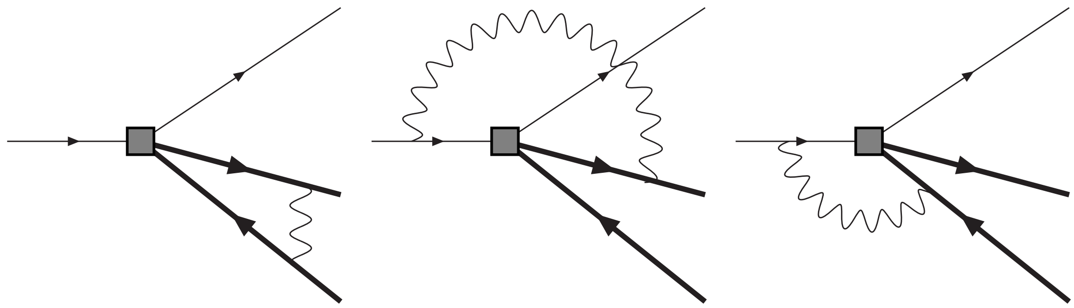
$$\Gamma_0 = \frac{G_F^2 M_l^5}{192\pi^3} N_c V_{ud}^2 \ . \tag{5}$$

At this order, Fermi theory and the Standard Model yield identical results.

2.2 One-loop electromagnetic corrections in Fermi theory

Let us turn to the one-loop corrections to this result. In the Standard Model, the decay rate Γ receives contributions from exchanges of virtual photons, weak gauge and Higgs bosons [10, 21]. Infrared divergences occur, but they are cancelled once we add the decay rate for real-photon emission $l \to \bar{u} + d + \nu + \gamma$. In Fermi theory, the one-loop corrections which involve two weak vertices are negligibly small and we only have to consider diagrams which involve the exchange of a photon between two charged fermion lines. This contribution contains infrared divergences, which will be cancelled by the decay rate for real-photon emission. At this order, the expression of the latter is identical to that in the Standard Model. In addition, starting at two loops (i.e. at order $O(\alpha\,\alpha_s)$) there appears QCD corrections to the decay amplitude. One can use Fermi theory whenever the momentum transferred by the virtual W boson is much smaller than its mass. One must also require that this momentum is sufficiently large as compared to 1 GeV such that perturbative QCD makes sense. For the moment, let us ignore these corrections. In sec. 3.5.2 we will discuss how to take them into account in an approximate way.

Therefore, we focus on $O(e^2)$ corrections to eq. (4) in Fermi theory caused by the exchange of a virtual photon. The Lagrangian which encodes the interactions of the



Q_2Q_3 $\qquad$ Q_0Q_2 $\qquad$ Q_0Q_3

Figure 2: One-loop electromagnetic corrections to the semi-leptonic decay of a lepton in Fermi Theory. Diagrams corresponding to wave-function renormalisation and proportional to Q_i^2 $(i = 0, 2, 3)$ are not shown.

photon field with the charged leptons and quarks is given by

$$\begin{aligned}\mathcal{L}_\gamma = \quad & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu F_\nu)^2 + \frac{1}{2}M_\gamma^2 F^\mu F_\mu + \\ & \bar{l}(i\not\partial - eQ_0 \not F - M_l)l + \bar{\nu}_L(i\not\partial)\nu_L + \sum_{q=u,d}\bar{q}(i\not\partial - eQ_q \not F)q\ . \end{aligned} \tag{6}$$

A small photon mass M_γ is introduced in order to control infrared divergences. In addition, we use the Pauli-Villars regularisation method to treat ultraviolet divergences. From the point of view of the chiral expansion the terms in eq. (6) have chiral order p^2 provided that counting rules are adopted

$$F_\mu \sim O(p^0) \qquad l,\ \nu_L,\ q\ \sim O(p^{\frac{1}{2}}) \qquad e,\ M_l,\ M_\gamma \sim O(p)\ . \tag{7}$$

In this paper, we will restrict ourselves to the Feynman gauge $\xi = 1$. Following BL's convention, we denote the charges of the lepton, the quark and the antiquark Q_0, Q_2 and Q_3 respectively. The physical values of these charges are

$$Q_0 = -1,\ Q_2 = -\frac{1}{3},\ Q_3 = -\frac{2}{3}\ . \tag{8}$$

Now, we determine the various contributions due to a virtual photon exchange, labeled in terms of these charges and shown in fig. 2.

2.2.1 Contributions Q_0^2 and $Q_2^2 + Q_3^2$

These contributions are given by the wave-function renormalisation. Including contributions up to one loop, the lepton propagator has the following form

$$G_F^l(p) = \frac{i}{\not p - M_l + \Sigma_l(p)} \tag{9}$$

with

$$\Sigma_l(p) = -Q_0^2 e^2 \int \frac{-id^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{p} + \not{k} + M_l)\gamma_\mu}{(k^2 - M_\gamma^2)_\Lambda((k+p)^2 - M_l^2)} \; . \tag{10}$$

The denominator $(k^2 - M_\gamma^2)_\Lambda$ stems from the photon propagator, regularised à la Pauli-Villars

$$\frac{1}{(k^2 - M_\gamma^2)_\Lambda} = \frac{-\Lambda^2}{(k^2 - M_\gamma^2)(k^2 - \Lambda^2)} \; . \tag{11}$$

The wave function renormalisation of the lepton requires to expand the lepton propagator around the mass-shell $p^2 = M_l^2$

$$G_F^l(p) \simeq \frac{i}{(1 + K_F^l)(\not{p} - M_l - \delta M_l)} \; . \tag{12}$$

A standard calculation gives

$$\begin{aligned} K_F^l &= \frac{-Q_0^2 e^2}{16\pi^2}\left(-\mathrm{div} + 2\log\frac{M_l}{\mu_0} - 4\log\frac{M_\gamma}{M_l} - \frac{9}{2}\right) \\ \delta M_l &= \frac{-Q_0^2 e^2}{16\pi^2} M_l \left(3\,\mathrm{div} - 6\log\frac{M_l}{\mu_0} + \frac{3}{2}\right) \; , \end{aligned} \tag{13}$$

with the (regularised) divergent piece

$$\mathrm{div} = \log\frac{\Lambda^2}{\mu_0^2} \; . \tag{14}$$

and μ_0 denotes the renormalisation scale in Fermi Theory. Applying the LSZ reduction formula (e.g. [22]) yields the correction of order $e^2 Q_0^2$ induced by the one-loop lepton propagator of the form (12)

$$T_{00} = T_0\left(-\frac{1}{2}K_l^F\right) = T_0 \times \frac{Q_0^2\alpha}{8\pi}\left(-\mathrm{div} + 2\log\frac{M_l}{\mu_0} - 4\log\frac{M_\gamma}{M_l} - \frac{9}{2}\right) \; . \tag{15}$$

Quark propagators are treated on the same footing apart from the fact that these fermions are assumed to be massless. In this case, one finds the wave-function renormalisation factor to be

$$K_F^q = \frac{-Q_q^2 e^2}{16\pi^2}\left(-\mathrm{div} + 2\log\frac{M_\gamma}{\mu_0}\right) \tag{16}$$

and the corresponding contribution to the decay amplitude reads

$$T_{qq} = T_0 \times (Q_2^2 + Q_3^2)\frac{\alpha}{8\pi}\left(-\mathrm{div} + 2\log\frac{M_\gamma}{\mu_0}\right) \; . \tag{17}$$

These yield the following corrections to the decay rate

$$\Gamma_{ii} = \Gamma_0 \frac{\alpha}{2\pi}\left[Q_0^2\left(-\frac{1}{2}\mathrm{div} + \log\frac{M_l}{\mu_0} - 2\log\frac{M_\gamma}{M_l} - \frac{9}{4}\right) + (Q_2^2 + Q_3^2)\left(-\frac{1}{2}\mathrm{div} + \log\frac{M_\gamma}{\mu_0}\right)\right] \tag{18}$$

2.2.2 Contribution Q_2Q_3

Here one considers the graph with one photon line joining the anti-quark u to the quark d (left-hand diagram in fig. 2). The amplitude has the form

$$T_{23} = -\frac{G_F V_{ud}}{\sqrt{2}} \bar{u}_\nu(p')\gamma_\lambda(1-\gamma^5)u_l(p)H_{23}^\lambda(q,q') \ , \tag{19}$$

where H_{23}^λ is given by

$$H_{23}^\lambda(q,q') = Q_2Q_3\, e^2\, \bar{u}_d(q')\gamma^\mu\gamma^\sigma\gamma^\lambda\gamma^\rho\gamma_\mu(1-\gamma^5)v_u(q) \int \frac{-id^4k}{(2\pi)^4} \frac{(k-q)_\rho(k+q')_\sigma}{D} \ , \tag{20}$$

with the denominator

$$D = (k^2 - M_\gamma^2)_\Lambda (k+q')^2(k-q)^2 \ . \tag{21}$$

Let us introduce the following notation for the various integrals

$$\begin{aligned}
&\int \frac{-id^4k}{(2\pi)^4}\frac{1}{D} = H(t) \ , \\
&\int \frac{-id^4k}{(2\pi)^4}\frac{k_\mu}{D} = H_0(t)(q_\mu - q'_\mu) \ , \\
&\int \frac{-id^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{D} = H_2(t)\, g_{\mu\nu} + H_3(t)(q_\mu q_\nu + q'_\mu q'_\nu) + H_4(t)(q_\mu q'_\nu + q_\nu q'_\mu) \ .
\end{aligned} \tag{22}$$

These integrals can be explicitly computed, yielding

$$\begin{aligned}
H(t) &= \frac{1}{16\pi^2}\left(\frac{1}{t}\log\frac{t}{M_\gamma^2}\log\frac{t+M_\gamma^2}{M_\gamma^2} + \frac{1}{t}\mathrm{dilog}\frac{t+M_\gamma^2}{M_\gamma^2}\right) \ , \\
H_0(t) &= \frac{1}{16\pi^2}\left(-\frac{1}{t} + \frac{1}{t}\log\frac{t}{M_\gamma^2}\right) \ , \\
H_2(t) &= \frac{1}{16\pi^2}\left(\frac{1}{4}\,\mathrm{div} - \frac{1}{4}\log\frac{t}{\mu_0^2} + \frac{3}{8}\right) \ , \\
H_4(t) &= \frac{1}{16\pi^2}\left(-\frac{1}{2t}\right) \ .
\end{aligned} \tag{23}$$

We note that $H_2(t)$ is the only integral which diverges as $\Lambda \to \infty$. Simplifying the Dirac structure leads to an amplitude proportional to the leading order one,

$$T_{23} = T_0 \times (-Q_2Q_3)\, e^2[4\,H_2 + 2t(-H + 2H_0 + H_4)] \ . \tag{24}$$

We keep only the terms which do not vanish when $M_\gamma \to 0$, and we obtain the decay width

$$\Gamma_{23} = \Gamma_0 \times Q_2Q_3\,\frac{\alpha}{2\pi}\left[-\mathrm{div} + \log\frac{M_l^2}{\mu_0^2} + 4\log^2\frac{M_\gamma}{M_l} + \frac{43}{3}\log\frac{M_\gamma}{M_l} + \frac{859}{72} - \frac{\pi^2}{3}\right] \ . \tag{25}$$

2.2.3 Contributions Q_0Q_2 and Q_0Q_3

Here we considers the diagrams with one photon line joining the lepton line to one of the quark lines. The contribution proportional to Q_0Q_2 (middle diagram in fig. 2) is given by

$$\begin{aligned} T_{02} = \quad & -\frac{G_F V_{ud}}{\sqrt{2}} Q_0 Q_2 e^2 \int \frac{-id^4k}{(2\pi)^4} \frac{1}{D_l} \times \\ & \bar{u}_d(q')\gamma^\alpha(\not{k} + \not{q}')\gamma^\lambda(1-\gamma^5)v_u(q)\, \bar{u}_\nu(p')\gamma_\lambda(1-\gamma^5)[\not{k} + \not{p} + M_l]\gamma_\alpha u_l(p) \ , \end{aligned} \tag{26}$$

with

$$D_l = (k^2 - M_\gamma^2)_\Lambda((k+p)^2 - M_l^2)(k+q')^2 \ . \tag{27}$$

As in the previous case, we introduce the various Feynman integrals

$$\begin{aligned} \int \frac{-id^4k}{(2\pi)^4} \frac{1}{D_l} &= I(u) \ , \\ \int \frac{-id^4k}{(2\pi)^4} \frac{k_\mu}{D_l} &= I_0(u)p_\mu + I_1(u)q'_\mu \ , \\ \int \frac{-id^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{D_l} &= I_2(u)g_{\mu\nu} + I_3(u)p_\mu p_\nu + I_4(u)(p_\mu q'_\nu + p_\nu q'_\mu) + I_5(u)q'_\mu q'_\nu \ . \end{aligned} \tag{28}$$

which can be computed easily

$$\begin{aligned} I(u) &= \frac{1}{16\pi^2} \frac{-1}{M_l^2 - u} \left[\left(\log\frac{M_\gamma}{M_l} - \log x_u \right)^2 + \mathrm{dilog}(x_u) + \frac{\pi^2}{4} \right] \ , \\ I_0(u) &= \frac{1}{16\pi^2} \left(-\frac{1}{u} \right) \log x_u \ , \\ I_1(u) &= \frac{1}{16\pi^2} \frac{1}{M_l^2 - u} \left[-2\log\frac{M_\gamma}{M_l} - 1 + \left(1 + \frac{M_l^2}{u} \right) \log x_u \right] \ , \\ I_2(u) &= \frac{1}{16\pi^2} \left(\frac{1}{4} \right) \left[\mathrm{div} - \log\frac{M_l^2}{\mu_0^2} + \frac{M_l^2 - u}{u} \log x_u + \frac{3}{2} \right] \ , \\ I_3(u) &= \frac{1}{16\pi^2} \left(-\frac{1}{2} \right) \left[\frac{1}{u} + \frac{M_l^2 - u}{u^2} \log x_u \right] \ , \\ I_4(u) &= \frac{1}{16\pi^2} \left(\frac{1}{2} \right) \left[\frac{1}{u} + \frac{M_l^2}{u^2} \log x_u \right] \ . \end{aligned} \tag{29}$$

with

$$x_u = \frac{M_l^2 - u}{M_l^2} \ . \tag{30}$$

All these integrals are convergent as $\Lambda \to \infty$ except I_2. Coming back to the amplitude, we simplify the Dirac algebra and end up with the following structure

$$T_{02} = -\frac{G_F}{\sqrt{2}} Q_0 Q_2 e^2 \Big\{ \bar{u}_d(q')\gamma^\nu(1-\gamma^5)v_u(q)\, \bar{u}_\nu(p')\gamma^\mu(1-\gamma^5)u_l(p)$$

$$\times\, 4[I_2(u) g_{\mu\nu} + p_\mu p_\nu (I_0(u) + I_3(u)) + q'_\mu p_\nu (I(u) + I_0(u) + I_1(u) + I_4(u))]$$
$$+\, \bar{u}_d(q')\gamma^\lambda (1-\gamma^5) v_u(q)\, \bar{u}_\nu(p')\gamma^\mu \gamma_\lambda (1+\gamma^5) u_l(p)$$
$$\left. \times\, (-2M_l)[I_0(u) p_\mu + (I(u) + I_1(u)) q'_\mu] \right\} . \tag{31}$$

The graph which gives the contribution proportional to $Q_0 Q_3$ is similar (right-hand diagram in fig. 2), but it involves the functions $I_i(s)$ instead of $I_i(u)$. In this case, the result reads

$$T_{03} = -\frac{G_F}{\sqrt{2}} Q_0 Q_3\, e^2 \Big\{ \bar{u}_d(q')\gamma^\lambda (1-\gamma^5) v_u(q)\, \bar{u}_\nu(p')\gamma_\lambda (1-\gamma^5) u_l(p)$$
$$\times\, 4[4\, I_2(s) + p^2 (I_0(s) + I_3(s)) + p.q (I(s) + I_0(s) + I_1(s) + 2 I_4(s))]$$
$$+\, \bar{u}_d(q')\gamma^\lambda (1-\gamma^5) v_u(q)\, \bar{u}_\nu(p')\gamma_\lambda \gamma^\mu (1+\gamma^5) u_l(p)$$
$$\left. \times\, (-2M_l)[I_0(s) p_\mu + (I(s) + I_1(s)) q_\mu] \right\} . \tag{32}$$

One notices that the divergent piece in T_{02} and in T_{03} is proportional to the leading-order amplitude T_0.

The resulting one-loop corrections to the decay rate are

$$\Gamma_{02} = \Gamma_0 Q_0 Q_2 \frac{\alpha}{2\pi} \left[\mathrm{div} - \log \frac{M_l^2}{\mu_0^2} - 2 \log^2 \frac{M_\gamma}{M_l} - \frac{17}{3} \log \frac{M_\gamma}{M_l} - \frac{5}{6}\pi^2 - \frac{1}{8} \right] . \tag{33}$$
$$\Gamma_{03} = \Gamma_0 Q_0 Q_3 \frac{\alpha}{2\pi} \left[4\, \mathrm{div} - 4 \log \frac{M_l^2}{\mu_0^2} - 2 \log^2 \frac{M_\gamma}{M_l} - \frac{19}{3} \log \frac{M_\gamma}{M_l} - \frac{5}{6}\pi^2 + \frac{47}{72} \right] .$$

This completes the calculation of the electromagnetic corrections at order e^2 to the decay amplitude (1) in Fermi theory. The result is ultraviolet divergent as well as infrared divergent. The latter divergence disappears upon adding to the decay rate the one involving a real photon, $\Gamma(l \to \bar{u} + d + \nu + \gamma)$ (its explicit expression can be found in ref. [21]). Ultraviolet divergences can be absorbed into local counterterms which we discuss in the next section.

2.3 Counterterms and matching

The previous calculations show that the one-loop ultraviolet divergences in the decay amplitude $T(p, q, p', q')$ are proportional to the leading-order amplitude $T_0(p, q, p', q')$. Thus, we may remove the divergences simply by adding a set of four counterterms proportional to the original Fermi Lagrangian

$$\mathcal{L}_{CT} = -\frac{4G_F V_{ud}}{\sqrt{2}} e^2 \left\{ \bar{l}_L \gamma^\lambda \nu_L \times \overline{d}_L \gamma_\lambda u_L + h.c. \right\}$$
$$\times \left[g_{00} Q_0^2 + g_{23} (Q_2 + Q_3)^2 + g_{02} Q_0 Q_2 + g_{03} (-Q_0 Q_3) \right] \ . \quad (34)$$

We will recast some terms into a more standard form soon. At this stage however, we just want to match the computation in the Fermi theory with that in the Standard Model. The decay amplitude can be made finite by imposing the following relations among bare and renormalised couplings in the Lagrangian (34)

$$\begin{aligned} g_{00} &= \frac{1}{16\pi^2}\left(\frac{1}{2}\log\frac{\Lambda^2}{\mu_0^2}\right) + g_{00}^r(\mu_0) \ , \\ g_{23} &= \frac{1}{16\pi^2}\left(\frac{1}{2}\log\frac{\Lambda^2}{\mu_0^2}\right) + g_{23}^r(\mu_0) \ , \\ g_{02} &= \frac{1}{16\pi^2}\left(-\log\frac{\Lambda^2}{\mu_0^2}\right) + g_{02}^r(\mu_0) \ , \\ g_{03} &= \frac{1}{16\pi^2}\left(4\log\frac{\Lambda^2}{\mu_0^2}\right) + g_{03}^r(\mu_0) \ , \end{aligned} \quad (35)$$

which leads to the renormalised one-loop correction to the decay rate in Fermi theory

$$\Gamma_{\text{Fermi}} = e^2\Gamma_0 \left\{ \frac{1}{16\pi^2}\left[6(1+\overline{Q})\log\frac{\mu_0}{M_l} + \overline{Q}\left(\frac{7}{4}+\frac{3}{4}\overline{Q}\right) + \frac{27}{2} - 2\pi^2\right] \right. \quad (36)$$
$$\left. + 2g_{00}^r(\mu_0) + 2g_{23}^r(\mu_0) + (1-\overline{Q})g_{02}^r(\mu_0) - (1+\overline{Q})g_{03}^r(\mu_0) \right\} ,$$

where the tree-level decay rate Γ_0 is given by eq. (5). The rate Γ_{Fermi} in eq. (36) also includes the process with a real photon in the final state and, as a consequence, there is no infrared divergence any more. In this expression, we have restricted the values of the electric charges (which were arbitrary up to now) to their physical values for Q_0 and the sum $Q_2 + Q_3 = Q_0$, but we have left arbitrary the difference

$$\overline{Q} = Q_2 - Q_3 \ . \quad (37)$$

One must then equate this expression for the decay rate to the Standard Model one, which reads according to ref. [21]

$$\Gamma_{SM} = \frac{\bar{\alpha}^2 M_l^5}{384\pi s_W^4 M_W^4}\left\{1 - 2\frac{\Pi_W(0)}{M_W^2} + \frac{\alpha}{2\pi}\left[3[1 - Q_0(Q_2 - Q_3)]\log\frac{M_Z}{M_l} \right.\right.$$
$$+ \left[\frac{7}{s_W^4} - \frac{4}{s_W^2}\right]\log c_W + \frac{6}{s_W^2} - \frac{1}{2} + \left[\frac{89}{24} - \pi^2\right]Q_0^2$$
$$\left.\left. - \frac{43}{24}(Q_2^2 + Q_3^2) + \frac{29}{6}Q_0Q_2 + \frac{237}{36}Q_0Q_3 - \frac{61}{12}Q_2Q_3\right]\right\} \quad (38)$$

where $\bar{\alpha}$ is the running QED coupling constant, c_W and s_W denote the cosine and sine of the Weinberg angle and $\Pi_W(0)$ is the W-propagator correction, renormalised on the mass shell and evaluated at zero momentum. This expression is valid provided that electric charge conservation holds, i.e. $Q_0 = Q_2 + Q_3$. The cancellation of ultraviolet divergences imposes that $Q_0^2 = 1$, while the difference $Q_2 - Q_3$ may be kept as a free parameter.

Before matching the expressions (36) and (38), we must express the combination of Standard Model parameters $\bar{\alpha}/(s_W^2 M_W^2)$ in terms of the Fermi coupling G_F. This relation is obtained by matching the expressions for the muon lifetime ($Q_0 = Q_2 = -1$ and $Q_3 = 0$) in both theories [21], including radiative corrections at one loop. Doing so provides the relation between the running QED coupling and the Fermi constant

$$\frac{\bar{\alpha}}{\sqrt{2} s_W^2 M_W^2} = \frac{G_F}{\pi}\left[1 + \frac{\Pi_W(0)}{M_W^2} + \frac{\alpha}{2 s_W^2 \pi}\left(\log c_W \left(2 - \frac{7}{2 s_W^2}\right) - 3\right)\right] . \tag{39}$$

Replacing in eq. (38) we can re-express BL's result in terms of G_F,

$$\Gamma_{SM} = \Gamma_0 \left\{1 + \frac{\alpha}{4\pi}\left[6(1+\overline{Q})\log\frac{M_Z}{M_l} + \overline{Q}\left(\frac{7}{4} + \frac{3}{4}\overline{Q}\right) + \frac{27}{2} - 2\pi^2\right]\right\} . \tag{40}$$

Since $\overline{Q}$ need not be set to its physical value, matching eq. (40) with eq. (36) generates two independent equations for the counterterm coupling constants

$$\begin{aligned} g_{02}^r(\mu_0) + g_{03}^r(\mu_0) &= \frac{1}{16\pi^2}\left[-6\log\frac{M_Z}{\mu_0}\right] , \\ g_{00}^r(\mu_0) + g_{23}^r(\mu_0) + g_{02}^r(\mu_0) &= 0 . \end{aligned} \tag{41}$$

This ends the first matching step.

3 Matching Fermi theory and the chiral Lagrangian at one loop

The second matching step proceeds in a rather different way from the first one. We will consider the effective chiral Lagrangian in which spurion sources are introduced for the purpose of classifying the independent terms. The trick will be to define correlators associated with these sources and to compute them in the two different effective theories.

3.1 Chiral Lagrangian with dynamical photons and leptons

Coupling QCD to electromagnetism breaks chiral symmetry explicitly because the quark charge matrix Q is not proportional to the unit matrix. Coupling to the weak interaction generates an additional breaking induced by the weak charge matrix Q_W. We will apply the chiral expansion to the three lightest quarks such that these matrices are

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} , \quad Q_W = -2\sqrt{2}\begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{42}$$

The neutral current part of the weak interaction will not be considered here. At the level of the effective Lagrangian, the symmetry breaking induced by Q and Q_W can be accounted for and treated perturbatively by using the spurion formalism. The treatment is analogous to the case of the symmetry breaking caused by the quark mass matrix $\mathcal{M}$. In that case, one replaces the physical mass matrix by a pair of sources $s(x)$, $p(x)$ to which one ascribes a transformation rule under the chiral group [13, 14]

$$s(x) + ip(x) \to g_R \,[s(x) + ip(x)]\, g_L^\dagger \ , \tag{43}$$

where (g_L, g_R) is a group element. In the same manner, one replaces the electric charge matrix Q by two spurion sources $\mathbf{q}_L(x)$, $\mathbf{q}_R(x)$ [16] and the weak charge matrix Q_W by one spurion source $\mathbf{q}_W(x)$ [17]. The part of the Lagrangian accounting for the coupling of the light quarks to the photon and to a lepton pair is then written as

$$\begin{aligned}\mathcal{L}^{spurions}_{QCD+Fermi} = &-e\, F_\lambda(\overline{\psi}_L \mathbf{q}_L \gamma^\lambda \psi_L + \overline{\psi}_R \mathbf{q}_R \gamma^\lambda \psi_R) \\ &- \frac{4G_F}{\sqrt{2}} \left(\bar{l}_L \gamma_\lambda \nu_L \, \overline{\psi}_L \mathbf{q}_W \gamma^\lambda \psi_L + \overline{\nu}_L \gamma_\lambda l_L \, \overline{\psi}_L \mathbf{q}_W^\dagger \gamma^\lambda \psi_L \right) \ ,\end{aligned} \tag{44}$$

where ψ collects the u, d, s quark fields. Chiral invariance is satisfied provided the spurion sources are assumed to transform as

$$\mathbf{q}_R(x) \to g_R \,\mathbf{q}_R(x)\, g_R^\dagger,\ \ \mathbf{q}_L(x) \to g_L \,\mathbf{q}_L(x)\, g_L^\dagger,\ \ \mathbf{q}_W(x) \to g_L \,\mathbf{q}_W(x)\, g_L^\dagger \ . \tag{45}$$

It is also convenient to endow the spurions with the chiral order

$$\mathbf{q}_L,\ \mathbf{q}_R,\ \mathbf{q}_W \ \sim O(p^0) \ . \tag{46}$$

Having defined the transformations of the spurion fields, one can build the most general effective Lagrangian satisfying chiral symmetry with pseudo-Goldstone bosons, photon and light leptons as dynamical fields. This Lagrangian provides a complete low-energy description of the Standard Model. We deal with massless quarks, which means that we take the chiral limit $m_u = m_d = m_s = 0$ (in the following, QCD in this limit will be called "chiral QCD" for concision).

The pseudo-Goldstone mesons (π, K, η) are included into a unitary matrix $U = u^2$ and a so-called "building block" u_μ (see e.g. [23])

$$u_\mu = iu^\dagger D_\mu U u^\dagger, \quad D_\mu U = \partial_\mu U - ir_\mu U + iU l_\mu \ , \tag{47}$$

where l_μ, r_μ contain not only vector and axial-vector sources for the corresponding QCD currents, but also the photon, the light leptons and the spurion sources,

$$\begin{aligned} l_\mu &= v_\mu - a_\mu - e\mathbf{q}_L F_\mu + G_F[\mathbf{q}_W l_L \gamma_\mu \nu_L + \mathbf{q}_W^\dagger \overline{\nu}_L \gamma_\mu l_L] \ , \\ r_\mu &= v_\mu + a_\mu - e\mathbf{q}_R F_\mu \ . \end{aligned} \tag{48}$$

Chiral "building blocks" may be constructed from the spurion fields

$$\mathcal{Q}_R \equiv u^\dagger \mathbf{q}_R u,\ \ \mathcal{Q}_L \equiv u \mathbf{q}_L u^\dagger,\ \ \mathcal{Q}_W \equiv u \mathbf{q}_W u^\dagger \ , \tag{49}$$

and also

$$\mathcal{Q}_R^\mu \equiv u^\dagger D^\mu \mathbf{q}_R u,\ \ \mathcal{Q}_L^\mu \equiv u D^\mu \mathbf{q}_L u^\dagger \ . \tag{50}$$

The covariant derivatives for the spurions are defined as [16]

$$D^\mu \mathbf{q}_R \equiv \partial^\mu \mathbf{q}_R - i[r_\mu, \mathbf{q}_R], \quad D^\mu \mathbf{q}_L \equiv \partial^\mu \mathbf{q}_L - i[l_\mu, \mathbf{q}_L] \ . \tag{51}$$

In ref. [16], the independent terms that contain one pair of spurions $\mathbf{q}_L$, $\mathbf{q}_R$ and that contribute up to $O(p^4)$ were classified. One of these terms will play a special role in our discussion, namely

$$\mathcal{L}^{(12)}_{\text{Urech}} = -ie^2 F_0^2 K_{12} \langle u_\mu ([\mathcal{Q}_L^\mu, \mathcal{Q}_L] - [\mathcal{Q}_R^\mu, \mathcal{Q}_R]) \rangle \ . \tag{52}$$

Knecht *et al.* [17] listed the $O(p^4)$ elements of the chiral Lagrangian that involve a light lepton pair and are associated with semi-leptonic decays. They obtained seven independent terms, once the following constraint was implemented

$$\mathbf{q}_L \mathbf{q}_W = \frac{2}{3}\mathbf{q}_W, \quad \mathbf{q}_W \mathbf{q}_L = -\frac{1}{3}\mathbf{q}_W \ . \tag{53}$$

Since we want to discuss the physical interpretation of the associated LEC's X_i, it is convenient to consider a somewhat more general situation and relax the constraint (53). This leads to an extended chiral Lagrangian which contains two additional terms. The associated LEC's will be called $\hat{X}_1$, $\hat{X}_2$. The remaining terms and the associated LEC's are identical to the case considered in ref. [17], except for the LEC's X_6 and X_7 which have different values in the two settings and which will be labelled $\hat{X}_6$, $\hat{X}_7$ in our case. The extended Lagrangian reads

$$\begin{aligned}
\mathcal{L}_{\text{leptons}} = e^2 \sum_l \Big\{ & F_0^2 G_F \Big[X_1 \bar{l}_L \gamma_\mu \nu_L \langle u^\mu \{\mathcal{Q}_R, \mathcal{Q}_W\}\rangle + \hat{X}_1 \bar{l}_L \gamma_\mu \nu_L \langle u^\mu \{\mathcal{Q}_L, \mathcal{Q}_W\}\rangle \\
& + X_2 \bar{l}_L \gamma_\mu \nu_L \langle u^\mu [\mathcal{Q}_R, \mathcal{Q}_W]\rangle + \hat{X}_2 \bar{l}_L \gamma_\mu \nu_L \langle u^\mu [\mathcal{Q}_L, \mathcal{Q}_W]\rangle \\
& + X_3 M_l \bar{l}_R \nu_L \langle \mathcal{Q}_R \mathcal{Q}_W \rangle \\
& + iX_4 \bar{l}_L \gamma_\mu \nu_L \langle \mathcal{Q}_L^\mu \mathcal{Q}_W \rangle + iX_5 \bar{l}_L \gamma_\mu \nu_L \langle \mathcal{Q}_R^\mu \mathcal{Q}_W \rangle + h.c. \Big] \\
& + \hat{X}_6 \bar{l}(i \not\partial + e \not{A}) l + \hat{X}_7 M_l \bar{l} l \Big\} \ .
\end{aligned} \tag{54}$$

An additional possible term of the form $M_l \bar{l}_R \nu_L \langle \mathbf{q}_L \mathbf{q}_W \rangle$ is of no practical relevance and will be ignored. The original LEC's X_6 and X_7 are easy to relate to the new set of LEC's $\hat{X}_i$

$$\begin{aligned}
X_6 &= \hat{X}_6 + \frac{4}{3}\hat{X}_1 + 4\hat{X}_2 \ , \\
X_7 &= \hat{X}_7 - \frac{4}{3}\hat{X}_1 - 4\hat{X}_2 \ .
\end{aligned} \tag{55}$$

These relations allow us to disentangle the strong-interaction content of X_6 and X_7, corresponding to $\hat{X}_1$ and $\hat{X}_2$, and the electroweak contributions encoded in $\hat{X}_6$ and $\hat{X}_7$.

Finally, let us make two remarks. Firstly, one can verify that the Lagrangian terms listed in (54) do have chiral order p^4 if one uses the counting rules (7) and (46). Secondly, eq. (54) obviously does not exhaust the possible terms of order p^4 involving light lepton pairs: they include only those connected with charged currents. Terms related with neutral currents are disregarded here (for some examples, see e.g. [24]).

3.2 Spurion correlators

We have included electric and weak charge spurion sources in the Lagrangian. Therefore, in addition to the usual vector, axial-vector... sources, the generating functional depends on $\mathbf{q}_L(x)$, $\mathbf{q}_R(x)$, $\mathbf{q}_W(x)$. We can define generalised Green functions by taking derivatives of the generating functional with respect to these sources, and eventually with respect to the usual sources, in order to compute matrix elements between physical states. This idea was used in ref. [18] to generate a set of sum rules obeyed by the LEC's K_i [16]. It can be extended to the present situation without difficulty, and we define a set of three matrix elements of three operators, obtained by taking one functional derivative with respect to an electric charge spurion and one derivative with respect to a weak charge spurion.

More specifically, we introduce the charge spurions $\mathbf{q}_V(x)$ and $\mathbf{q}_A(x)$ as

$$\mathbf{q}_L(x) = \frac{1}{2}(\mathbf{q}_V(x) - \mathbf{q}_A(x)), \quad \mathbf{q}_R(x) = \frac{1}{2}(\mathbf{q}_V(x) + \mathbf{q}_A(x)) \ . \tag{56}$$

and the correlators

$$i \int d^4x \, \mathrm{e}^{irx} \langle l(p)\bar{\nu}(q)| \frac{\delta^2 W(\mathbf{q}_L, \mathbf{q}_R, \mathbf{q}_W)}{\delta \mathbf{q}_{R^b}(x) \delta \mathbf{q}_{W^c}(0)} |0\rangle \equiv \delta^{bc} G_{RW}(p,q,r) \ . \tag{57}$$

and

$$\begin{aligned} \int d^4x \, \langle l(p)\bar{\nu}(q)| \frac{\delta^2 W(\mathbf{q}_V, \mathbf{q}_A, \mathbf{q}_W)}{\delta \mathbf{q}_{V^b}(x) \delta \mathbf{q}_{W^c}(0)} |\pi^a(r)\rangle &\equiv i f^{abc} F_{VW}(p,q,r) + d^{abc} D_{VW}(p,q,r) \\ \int d^4x \, \langle l(p)\bar{\nu}(q)| \frac{\delta^2 W(\mathbf{q}_V, \mathbf{q}_A, \mathbf{q}_W)}{\delta \mathbf{q}_{A^b}(x) \delta \mathbf{q}_{W^c}(0)} |\pi^a(r)\rangle &\equiv i f^{abc} F_{AW}(p,q,r) + d^{abc} D_{AW}(p,q,r) \end{aligned} \tag{58}$$

where f_{abc} and d_{abc} denote the standard antisymmetric and symmetric functions defined through the commutation and anticommutation of Gell-Mann matrices. Once the functional derivatives have been taken, we set all sources to zero (including the charge spurions).

In the following, we will compute these generalised correlators in two different ways: firstly from the chiral Lagrangian, leading to expressions in terms of low-energy coupling constants, and secondly from the QCD and Fermi Lagrangians, yielding the correlators in terms of the counterterms in Fermi theory. This approach allows one to generate representations of the chiral coupling constants in terms of pure QCD correlation functions in a rather straightforward way, with a clear identification of the short-distance contributions from the Standard Model.

3.3 Correlators from the chiral Lagrangian at one loop

Let us start with the chiral Lagrangian. The spurion correlators receive tree-level contributions from $O(p^4)$ LEC's, and one-loop contributions with $O(p^2)$ vertices. Let us illustrate this in the case of $G_{RW}(p,q,r)$. The tree contribution involves X_3 and X_5

$$G_{RW}^{\rm tree}(p,q,r) = \frac{1}{2}e^2 G_F F_0^2 \Big[M_l X_3 \bar{u}_l(p)\frac{1-\gamma_5}{2}v_\nu(q) - X_5 u_l(p)\gamma_\mu \frac{1-\gamma_5}{2}v_\nu(q) r^\mu \Big] \ . \qquad (59)$$

The one-loop contribution has the following expression

$$G_{RW}^{\rm loop} = -\frac{e^2 Q_0 G_F F_0^2}{4}\int \frac{-id^dk}{(2\pi)^d}\left[\frac{(k+r)^\sigma (k+r)^\lambda}{(k+r)^2} - g^{\sigma\lambda}\right] \qquad (60)$$

$$\frac{1}{(k^2-M_\gamma^2)((k-p)^2-M_l^2)}\bar{u}_l(p)\gamma_\sigma(\not{p}-\not{k}+M_l)\gamma_\lambda\frac{1-\gamma_5}{2}v_\nu(q) \ .$$

In this sector, the ultraviolet divergences will be controlled via dimensional regularisation (as usual in chiral perturbation theory). We must compute $G_{RW}^{\rm loop}$ only up to $O(r)$. This means that we may expand the integral for small values of the pion momentum r up to linear order. A further simplification consists in expanding in powers of the lepton mass M_l around the limit $M_l = 0$, keeping $M_\gamma \neq 0$ whenever necessary to avoid infrared divergences. After performing these expansions, the loop contribution exhibits the following explicit expression

$$G_{RW}^{\rm loop}(p,q,r) = -e^2 Q_0 G_F F_0^2 \Bigg\{ M_l \bar{u}_l(p)\frac{1-\gamma_5}{2}v_\nu(q)\left[\frac{3}{2}{\rm div}_\chi + \frac{1}{16\pi^2}\left(\frac{3}{4}\log\frac{M_\gamma^2}{\mu^2}+\frac{1}{8}\right)\right]$$
$$+\bar{u}_l(p)\gamma_\mu\frac{1-\gamma_5}{2}v_\nu(q)r^\mu\left[\frac{3}{4}{\rm div}_\chi + \frac{1}{16\pi^2}\left(\frac{3}{8}\log\frac{M_\gamma^2}{\mu^2}+\frac{1}{16}\right)\right]\Bigg\} \ , \qquad (61)$$

where the chiral divergence is defined in the customary way

$${\rm div}_\chi = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}\left(\log 4\pi - \gamma + 1\right)\right\} \ . \qquad (62)$$

The complete chiral expression for G_{RW} is obtained by adding tree (59) and one-loop (61) pieces

$$G_{RW}^{\rm chir}(p,q,r) = G_{RW}^{\rm tree}(p,q,r) + G_{RW}^{\rm loop}(p,q,r) \ . \qquad (63)$$

The ultraviolet divergences are absorbed into the LEC's

$$X_i = X_i^r(\mu) + \Xi_i\,\frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4} - \frac{1}{2}(\log 4\pi - \gamma + 1)\right) \ . \qquad (64)$$

This requirement sets the divergence coefficients

$$\Xi_3 = -3 \ , \qquad \Xi_5 = \frac{3}{2} \ , \qquad (65)$$

in agreement with ref. [17]. We proceed in exactly the same way with the other spurion correlators F_{VW}, D_{VW}, F_{AW}, D_{AW} (the loop contribution to D-terms involve correlators that vanish by invariance under charge conjugation). These correlators, expanded up to linear order in the momentum r, have the following expressions at next-to-leading order

$$\begin{aligned}
F_{VW}^{\rm chir}(p,q,r) &= F_{VW}^{\rm chir}(p,q,0) + e^2 G_F F_0\, \bar{u}_l(p)\gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu \\
&\quad\times \left[X_2^r + \hat{X}_2^r + \frac{1}{16\pi^2}\left(\frac{5}{4}\log\frac{M_\gamma^2}{\mu^2} + \frac{1}{8} \right) \right] , \\
F_{AW}^{\rm chir}(p,q,r) &= F_{AW}^{\rm chir}(p,q,0) + e^2 G_F F_0\, \bar{u}_l(p)\gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu \\
&\quad\times \left[X_2^r - \hat{X}_2^r + \frac{1}{16\pi^2}\left(-\frac{1}{2}\log\frac{M_\gamma^2}{\mu^2} \right) \right] , \\
D_{VW}^{\rm chir}(p,q,r) &= e^2 G_F F_0\, \bar{u}_l(p)\gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu [X_1 + \hat{X}_1] , \\
D_{AW}^{\rm chir}(p,q,r) &= e^2 G_F F_0\, \bar{u}_l(p)\gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu [X_1 - \hat{X}_1] . \qquad (66)
\end{aligned}$$

We have not written the explicit formulas for $F_{VW}^{\rm chir}(p,q,0)$ and $F_{AW}^{\rm chir}(p,q,0)$. The following simple relation holds

$$F_{VW}^{\rm chir}(p,q,0) = F_{AW}^{\rm chir}(p,q,0) = G_{RW}^{\rm chir}(p,q) , \qquad (67)$$

as a result of a soft-pion theorem (see eq. (85) below). The coefficients of the chiral divergences are

$$\Xi_1 = \hat{\Xi}_1 = 0 , \qquad \Xi_2 = -\frac{3}{4} , \qquad \hat{\Xi}_2 = -\frac{7}{4} , \qquad (68)$$

also in agreement with ref. [17].

3.4 Correlators from QCD + Fermi theory

Here we compute the correlators introduced in sec. 3.2 using the QCD and Fermi Lagrangians. A first (non-local) contribution stems from the terms in these Lagrangians which are linear in the spurion sources, see eq. (44). A second (local) contribution is due to the counterterms in Fermi theory that are quadratic in the spurions.

3.4.1 Integral contributions

Let us first consider the contribution to the spurion correlators coming from the Lagrangian eq. (44). Matrix elements of vector and axial-vector currents appear by taking the functional derivatives defining the spurion correlators. Let us introduce the following notation for these objects,

$$i\int d^4x\, {\rm e}^{ikx} \langle 0| V_\sigma^b(x) V_\lambda^c(0) - A_\sigma^b(x) A_\lambda^c(0) |0\rangle \equiv \delta^{bc}\Pi_{VV-AA}^{\sigma\lambda}(k) ,$$

$$\int d^4x\, \mathrm{e}^{ikx} \langle 0|V_\sigma^b(x) V_\lambda^c(0)|\pi^a(r)\rangle \equiv d^{abc}\Gamma_{VV}^{\sigma\lambda}(k,r) \ ,$$
$$\int d^4x\, \mathrm{e}^{ikx} \langle 0|A_\sigma^b(x) A_\lambda^c(0)|\pi^a(r)\rangle \equiv d^{abc}\Gamma_{AA}^{\sigma\lambda}(k,r) \ ,$$
$$\int d^4x\, \mathrm{e}^{ikx} \langle 0|V_\sigma^b(x) A_\lambda^c(0)|\pi^a(r)\rangle \equiv i f^{abc}\Gamma_{VA}^{\sigma\lambda}(k,r) \ . \tag{69}$$

The choice between the f^{abc} and the d^{abc} tensor in these equations is dictated by invariance under charge conjugation. Let us remark that the Pauli-Villars regularisation offers a very appealing feature here: the operators and matrix elements appearing in eqs. (69) are not to be defined in an arbitrary number of dimensions, but only in the physical (four-dimensional) case. The Lagrangian eq. (44) leads to contributions to the spurion correlators that are integrals involving the QCD Green function and vertex operators introduced above (69).

$$G_{RW}^{\mathrm{int}}(p,q,r) = -\frac{e^2 Q_0 G_F}{4} \int \frac{-id^4k}{(2\pi)^4} \Pi_{VV-AA}^{\sigma\lambda}(k+r) \times K_{\sigma\lambda}(k,p,q) \ , \tag{70}$$

with

$$K_{\sigma\lambda}(k,p,q) = \frac{1}{(k^2-M_\gamma^2)_\Lambda((k-p)^2-M_l^2)} \bar{u}_l(p)\gamma_\sigma(\not p - \not k + M_l)\gamma_\lambda \frac{1-\gamma_5}{2} v_\nu(q) \ , \tag{71}$$

and

$$F_{VW}^{\mathrm{int}}(p,q,r) = \frac{e^2 Q_0 G_F}{2} \int \frac{-id^4k}{(2\pi)^4} \Gamma_{VA}^{\sigma\lambda}(k,r) \times K_{\sigma\lambda}(k,p,q) \ ,$$
$$D_{VW}^{\mathrm{int}}(p,q,r) = -\frac{e^2 Q_0 G_F}{2} \int \frac{-id^4k}{(2\pi)^4} \Gamma_{VV}^{\sigma\lambda}(k,r) \times K_{\sigma\lambda}(k,p,q) \ ,$$
$$F_{AW}^{\mathrm{int}}(p,q,r) = \frac{e^2 Q_0 G_F}{2} \int \frac{-id^4k}{(2\pi)^4} \Gamma_{VA}^{\lambda\sigma}(r-k,r) \times K_{\sigma\lambda}(k,p,q) \ ,$$
$$D_{AW}^{\mathrm{int}}(p,q,r) = \frac{e^2 Q_0 G_F}{2} \int \frac{-id^4k}{(2\pi)^4} \Gamma_{AA}^{\sigma\lambda}(k,r) \times K_{\sigma\lambda}(k,p,q) \ . \tag{72}$$

The integral in eq. (70) converges when the Pauli-Villars regulator mass Λ is sent to infinity (there is no ultraviolet divergence), whereas the other integrals would diverge. The divergences will be removed upon adding the contributions generated from the Fermi counterterms.

3.4.2 Counterterm contributions

In order to identify the contributions to the spurion correlators arising from the Fermi counterterms (34), we must first rewrite the Lagrangian in terms of spurion sources. After some manipulations, we can re-express the counterterms as follows

$$\mathcal{L}_{\mathrm{CT}} = -2e^2 Q_0^2 g_{00}\, \bar{l}(i \not\partial - eQ_0 \not F - M_l) l$$

$$
\begin{aligned}
& - ie^2 g_{23} \left(\overline{\psi}_L [\mathbf{q}_L, D^\mu \mathbf{q}_L] \gamma_\mu \psi_L + L \leftrightarrow R \right) && (73) \\
& + e^2 Q_0 G_F \left\{ \bar{l}_L \gamma_\lambda \nu_L \times \left[g_{02}\, \overline{\psi}_L \gamma^\lambda\, \mathbf{q}_W \mathbf{q}_L \psi_L + g_{03}\, \overline{\psi}_L \gamma^\lambda\, \mathbf{q}_L \mathbf{q}_W \psi_L \right] + h.c. \right\} .
\end{aligned}
$$

The term proportional to g_{00} has been written in a more conventional way, which is equivalent to the formulation in eq. (34) as far as the amplitude T is concerned (we have applied equations of motion). We have extended the term proportional to g_{23} to comply with the transformation laws of the spurions: this extended term contains the piece proportional to g_{23} in the original formulation (34), as can be seen from the definition of the spurion derivative (51). Modulo these transformations, it is simple to check that setting the spurions to the physical charges $\mathbf{q}_L = \mathbf{q}_R = Q$, $\mathbf{q}_W = Q_W$ reproduces the Lagrangian eq. (34). Up to terms which are physically irrelevant, the translation from charge labels to spurions is essentially unique.

In this new form, it is an easy task to compute the functional derivatives and deduce the contributions to the spurion correlators. The following results are obtained

$$
\begin{aligned}
G^{\mathrm{CT}}_{RW} &= 0 \ , \\
F^{\mathrm{CT}}_{VW} &= e^2 G_F Q_0 F_0 \bar{u}_l(p) \gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu \left[\frac{1}{4} g_{02} - \frac{1}{4} g_{03} \right] \ , \\
D^{\mathrm{CT}}_{VW} &= e^2 G_F Q_0 F_0 \bar{u}_l(p) \gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu \left[-\frac{1}{4} g_{02} - \frac{1}{4} g_{03} \right] \ , \\
F^{\mathrm{CT}}_{AW} &= e^2 G_F Q_0 F_0 \bar{u}_l(p) \gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu \left[-\frac{1}{4} g_{02} + \frac{1}{4} g_{03} \right] \ , \\
D^{\mathrm{CT}}_{AW} &= e^2 G_F Q_0 F_0 \bar{u}_l(p) \gamma_\mu \frac{1-\gamma_5}{2} v_\nu(q) r^\mu \left[+\frac{1}{4} g_{02} + \frac{1}{4} g_{03} \right] \ . && (74)
\end{aligned}
$$

We can now add these contributions to the integral contributions

$$
G^{\mathrm{Fermi}}_{RW}(p,q,r) = G^{\mathrm{CT}}_{RW}(p,q,r) + G^{\mathrm{int}}_{RW}(p,q,r) \qquad (75)
$$

and similarly for the other correlators. The result should be finite as $\Lambda \to \infty$: we verify this now and show that the integrals can be brought to fairly simple forms.

3.5 Explicit representations of the chiral coupling constants

3.5.1 Integral representations

We can match the two expressions for the spurion correlators: the integral representation stemming from Fermi theory, such as eq. (75), and the formulae obtained from the chiral effective Lagrangian, see eq. (63). To do so, let us expand the integral representations discussed in sec. 3.4.1 for small values of the pion momentum r, and compare the series with the chiral expansion derived in sec. 3.3. This comparison yields integral representations for the LEC's of the chiral Lagrangian, which can be simplified further by displaying the kinematical structures and the associated form factors of the correlators involved. Let us first introduce the correlators related to G_{RW}, D_{VW} and D_{AW}

$$
\begin{aligned}
&\Pi^{\rho\sigma}_{VV-AA}(k) \equiv F_0^2(k^\rho k^\sigma - k^2 g^{\rho\sigma})\,\Pi_{VV-AA}(k^2) \ , \\
&\Gamma^{\rho\sigma}_{VV}(k,r) = iF_0\epsilon^{\rho\sigma\alpha\beta}k_\alpha r_\beta\,\Gamma_{VV}(k^2,k.r) \ , \\
&\Gamma^{\rho\sigma}_{AA}(k,r) = iF_0\epsilon^{\rho\sigma\alpha\beta}k_\alpha r_\beta\,\Gamma_{AA}(k^2,k.r) \ .
\end{aligned} \tag{76}
$$

In practice, we need $\Gamma^{\rho\sigma}_{VV}$ and $\Gamma^{\rho\sigma}_{AA}$ only up to $O(r)$, and thus it is enough to get the form factors $\Gamma_{VV}(k^2,k.r)$ and $\Gamma_{AA}(k^2,k.r)$ in the limit where the pion momentum r is set to zero. We use the simplified notation

$$
\lim_{r\to 0}\Gamma_{VV}(k^2,k.r) \equiv \Gamma_{VV}(k^2) \ , \qquad \lim_{r\to 0}\Gamma_{AA}(k^2,k.r) \equiv \Gamma_{AA}(k^2) \ . \tag{77}
$$

Then, in connection with the spurion correlators G_{RW}, D_{VW} and D_{AW}, we can obtain representations for the four LEC's X_1, $\hat{X}_1$, X_3 and X_5

$$
\begin{aligned}
X_1 &= -\frac{3}{8}\int\frac{-id^4k}{(2\pi)^4}\,\frac{1}{k^2}(\Gamma_{VV}(k^2)-\Gamma_{AA}(k^2)) \ , \\
\hat{X}_1 &= -\frac{3}{8}\int\frac{-id^4k}{(2\pi)^4}\,\frac{1}{k^2}\left(\Gamma_{VV}(k^2)+\Gamma_{AA}(k^2)-\frac{2}{k^2-\mu_1^2}\right)+\frac{3}{4}\log\frac{\mu_1^2}{M_Z^2} \ , \\
X_3^r(\mu) &= -\frac{3}{2}\int\frac{-id^4k}{(2\pi)^4}\,\frac{1}{k^2}\left(\Pi_{VV-AA}(k^2)+\frac{\mu_1^2}{k^2(k^2-\mu_1^2)}\right)+\frac{1}{16\pi^2}\left(\frac{3}{2}\log\frac{\mu^2}{\mu_1^2}-\frac{1}{4}\right) \ , \\
X_5^r(\mu) &= \frac{3}{4}\int\frac{-id^4k}{(2\pi)^4}\,\frac{1}{k^2}\left(\Pi_{VV-AA}(k^2)+\frac{\mu_1^2}{k^2(k^2-\mu_1^2)}\right) \\
&\qquad\qquad\qquad\qquad\qquad +\frac{1}{16\pi^2}\left(-\frac{3}{4}\log\frac{\mu^2}{\mu_1^2}-\frac{5}{8}\right) \ .
\end{aligned} \tag{78}
$$

These integrals could be rewritten as one-dimensional integrals. In order to derive the expression of $\hat{X}_1$, we have re-expressed the combination of counterterms $g_{02}^r(\mu_0)+g_{03}^r(\mu_0)$ using the matching conditions eqs. (41). The result involves an explicitly convergent integral, as can be checked easily using the asymptotic behaviour (see e.g. [25]) of $\Gamma_{VV}(k^2)$, $\Gamma_{AA}(k^2)$,

$$
\Gamma_{VV}(k^2),\ \Gamma_{AA}(k^2) \sim \frac{1}{k^2} \ , \tag{79}
$$

still ignoring (for the moment) perturbative QCD corrections. The scale μ_0, related to the renormalisation in Fermi theory, has disappeared, which signals that the original divergence was correctly cancelled by the counterterm. An arbitrary scale μ_1 has been introduced in the integrand to obtain convergent integrals, but the dependence on μ_1 cancels in the final result.

The integrals involved in X_1, X_2, X_5 converge because of the short-distance smoothness of the difference $VV-AA$ in chiral QCD (see e.g. [26]). In the case of X_3 and X_5, the integrands have been recast in a form which is explicitly infrared finite. As in the previous case, the overall results are independent of the scale μ_1 introduced in the integrands.

Similar sum rules can be written for X_2 and $\hat{X}_2$ by focusing on the r-linear piece in F_{AW} and F_{VW}. The function of interest is the vertex correlator $\Gamma^{\sigma\lambda}_{VA}(p,r)$ which involves two form factors F and G in the chiral limit [18],

$$\Gamma^{\sigma\lambda}_{VA}(p,r) = F_0 \left\{ \frac{(p^\sigma + 2q^\sigma)q^\lambda}{q^2} - g^{\sigma\lambda} + F(p^2,q^2)\, P^{\sigma\lambda} + G(p^2,q^2)\, Q^{\sigma\lambda} \right\} , \tag{80}$$

with $q = r - p$ and

$$P^{\sigma\lambda} = q^\sigma p^\lambda - (p.q)\, g^{\sigma\lambda}, \quad Q^{\sigma\lambda} = p^2 q^\sigma q^\lambda + q^2 p^\sigma p^\lambda - (p.q)\, p^\sigma q^\lambda - p^2 q^2 g^{\sigma\lambda} . \tag{81}$$

In order to identify the LEC's X_2 and $\hat{X}_2$ one must expand $\Gamma^{\sigma\lambda}_{VA}(p,r)$ in eqs. (72) up to linear order in the pion momentum r. Let us introduce

$$\begin{aligned} &f(k^2) \equiv F(k^2,k^2),\ f_1(k^2) \equiv \partial_x F(x,k^2)|_{x=k^2},\ f_2(k^2) \equiv \partial_y F(k^2,y)|_{y=k^2} \\ &g(k^2) \equiv G(k^2,k^2),\ g_1(k^2) \equiv \partial_x G(x,k^2)|_{x=k^2},\ g_2(k^2) \equiv \partial_y G(k^2,y)|_{y=k^2} \end{aligned} \tag{82}$$

The correct QCD asymptotic behaviour of $\Gamma^{\sigma\lambda}_{VA}(k,r)$ as $k \to \infty$ (see [18]) is reproduced up to order $1/k^2$ provided that these functions obey the limits

$$\lim_{k^2\to\infty} k^4 g(k^2) = -1,\ \lim_{k^2\to\infty} k^4 f(k^2) = \text{const.}\ ,\ \lim_{k^2\to\infty} k^4 (f_2(k^2) - k^2 g_2(k^2)) = -\frac{3}{2} . \tag{83}$$

After some quick algebra, we find the following integral representations for X_2 and $\hat{X}_2$ (once again essentially one-dimensional)

$$\begin{aligned} X_2^r(\mu) &= -\frac{3}{8}\int \frac{-id^4k}{(2\pi)^4}\frac{1}{k^2}\left(\Pi_{VV-AA}(k^2) + \frac{\mu_1^2}{k^2(k^2-\mu_1^2)}\right) + \frac{1}{16\pi^2}\left(\frac{3}{8}\log\frac{\mu^2}{\mu_1^2} + \frac{5}{16}\right) , \\ \hat{X}_2^r(\mu) &= -\frac{3}{8}\int \frac{-id^4k}{(2\pi)^4}\left[\frac{-1}{k^2(k^2-\mu_1^2)} + f_1(k^2) - f_2(k^2) + k^2\left(-g_1(k^2) + g_2(k^2)\right)\right] \\ &\quad + \frac{1}{16\pi^2}\left(-\frac{5}{4}\log\frac{\mu_0^2}{\mu_1^2} + \frac{7}{8}\log\frac{\mu^2}{\mu_1^2} - \frac{1}{16}\right) - \frac{1}{4}g_{02}^r(\mu_0) + \frac{1}{4}g_{03}^r(\mu_0) . \end{aligned} \tag{84}$$

In order to derive these expressions we have used integration by parts, noting that $f_1+f_2 = f'$ and $g_1 + g_2 = g'$, as well as the soft-pion theorem [27]

$$\Gamma^{\sigma\lambda}_{VA}(k,0) = \frac{1}{F_0}\Pi^{\sigma\lambda}_{VV-AA}(k) . \tag{85}$$

which implies that

$$\Pi_{VV-AA}(k^2) = \frac{1}{k^2} - f(k^2) + k^2 g(k^2) . \tag{86}$$

Let us remark that this soft-pion theorem, in combination with eq. (67), implies that the $O(r^0)$ pieces in F_{AW} and F_{VW} yield exactly the same sum rules as G_{RW}. One easily checks that the integral appearing in the expression of $\hat{X}_2^r$ is convergent whenever the form factors satisfy the QCD asymptotic constraints eqs. (83). Remarkably, the LEC X_2 turns out to depend only on the Green function $\langle VV - AA\rangle$. As in eq. (78), a scale μ_1

was introduced but the result is independent of μ_1. The result can also be verified to be independent of the scale μ_0 which is a consequence of the fact that the contribution from the counterterms correctly cancels the original divergence of the integral.

These exact integral representations reveal relationships among the coupling constants which were not a priori expected

$$\begin{aligned} X_3^r(\mu) &= 4X_2^r(\mu) - \frac{3}{2}\frac{1}{16\pi^2} \ , \\ X_5^r(\mu) &= -2X_2^r(\mu) \ . \end{aligned} \tag{87}$$

Let us emphasise that these relations are absolutely general. In particular, their validity is completely independent of any particular model for the two- and three-point Green functions involved in the integral representations.

3.5.2 The case of X_6

Among the LEC's which are physically relevant, X_6 plays a special role. According to eq. (55), X_6 can be expressed in terms of $\hat{X}_1$ and $\hat{X}_2$, which were discussed above, and $\hat{X}_6$. By construction, $\hat{X}_6$ has no strong-interaction content: it can be determined by computing the lepton wave-function renormalisation factor K_F in chiral perturbation theory and identifying it with our calculation in Fermi Theory in sec.2.2.1. The regularisation schemes are different: the former employs dimensional regularisation and chiral $\overline{MS}$ renormalisation, whereas the latter relies on Pauli-Villars regularisation. We get the relation

$$\hat{X}_6(\mu_0) = -2\, g_{00}^r(\mu_0) + \frac{3}{2}\frac{1}{16\pi^2} \ . \tag{88}$$

The resulting expression for X_6 involves a combination of counterterms, $-2g_{00} - g_{02} + g_{03}$, which is not determined by the matching conditions (41). This implies that physical quantities must involve X_6 together with one additional, electromagnetic, LEC. It is not difficult to see that this LEC must be K_{12}. We will see that the physically relevant combination is

$$X_6^{\text{phys}}(\mu) \equiv X_6^r(\mu) - 4K_{12}^r(\mu) = 4\,(\hat{X}_2^r(\mu) - K_{12}^r(\mu)) + \hat{X}_6^r(\mu) + \frac{4}{3}\hat{X}_1^r(\mu) \ . \tag{89}$$

The LEC K_{12} was shown to satisfy an integral representation in terms of the vertex function $\Gamma_{VA}^{\sigma\lambda}(k,r)$ that we have introduced above [18]. Using the present notation and regularisation scheme, one can derive the explicit representation

$$\begin{aligned} K_{12}^r(\mu) = &-\frac{3}{8}\int \frac{-id^4k}{(2\pi)^4}\left[\frac{-1}{k^2(k^2-\mu_1^2)} + f_1(k^2) - f_2(k^2) + k^2\left(-g_1(k^2) + g_2(k^2)\right)\right] \\ &+ \frac{1}{16\pi^2}\left(-\frac{1}{4}\log\frac{\mu_0^2}{\mu_1^2} - \frac{1}{8}\log\frac{\mu^2}{\mu_1^2} - \frac{5}{16}\right) + \frac{1}{2}\, g_{23}^r(\mu_0) \end{aligned} \tag{90}$$

where integration by parts was used to simplify the formula.

Let us now consider the difference $\hat{X}_2 - K_{12}^r$. Using the integral expressions (90) and (84), one observes that all terms cancel except for the counterterms

$$\hat{X}_2^r(\mu) - K_{12}^r(\mu) = -\frac{1}{4}g_{02}^r(\mu_0) + \frac{1}{4}g_{03}^r(\mu_0) - \frac{1}{2}g_{23}^r(\mu_0) + \frac{1}{16\pi^2}\left(-\log\frac{\mu_0^2}{\mu^2} + \frac{1}{4}\right) . \quad (91)$$

Inserting this result into X_6^{phys} and setting $\mu_0 = \mu$, one realises that the resulting combination of counterterms is indeed determined from the matching conditions (41)

$$-g_{02}^r(\mu) + g_{03}^r(\mu) - 2g_{00}^r(\mu) - 2g_{23}^r(\mu) = \frac{1}{16\pi^2}\left(-6\log\frac{M_Z}{\mu}\right) . \quad (92)$$

One ends up with the following simple representation of X_6^{phys}

$$\begin{aligned} X_6^r(\mu) - 4K_{12}^r(\mu) = & -\frac{1}{2}\int\frac{-id^4k}{(2\pi)^4}\frac{1}{k^2}\left(\Gamma_{VV}(k^2) + \Gamma_{AA}(k^2) - \frac{2}{k^2-\mu_1^2}\right) \\ & + \frac{1}{16\pi^2}\left[-8\log\frac{M_Z}{\mu_1} + 3\log\frac{\mu^2}{\mu_1^2} + \frac{5}{2}\right] . \end{aligned} \quad (93)$$

One can verify that in the calculations of radiative corrections currently available [17, 7, 28] X_6 and K_{12} are always involved through the above combination. This contribution, related to wave-function renormalisation, has the property of being universal, i.e., it appears as a multiplicative factor

$$S_{EW} = 1 - \frac{1}{2}e^2(X_6^r - 4K_{12}^r) , \quad (94)$$

in front of the amplitude independently of the specific process considered. We recover here the universal logarithmically enhanced $\log M_Z$ term identified by Sirlin [11], but we also get an explicit expression for the remaining terms.

Let us now consider the problem of perturbative QCD contributions. The couplings X_i which are related to the difference $VV - AA$, clearly, will be essentially unaffected by these. On the contrary, the combination X_6^{eff} is concerned by such corrections. In fact, if we take into account the correction proportional to α_s in the asymptotic behaviour of $\Gamma_{VV} + \Gamma_{AA}$ [10]

$$(\Gamma_{VV}(k^2) + \Gamma_{AA}(k^2))_{\alpha_s} \sim -\frac{2}{\pi}\frac{\alpha_s(k^2)}{k^2} , \quad (95)$$

in eq. (93) the integral will diverge. This is to be expected since the counterterms proportional to $\alpha\,\alpha_s$ have not been implemented. Inspired by the work of Sirlin [11], one can rather easily surmount this difficulty. The key point is that the asymptotic behaviour of $\Gamma_{VV} + \Gamma_{AA}$ is expected to set in at a scale μ_2 which is much smaller than M_Z. As a consequence, we can rewrite eq. (93), up to very small corrections of order $(\mu_2/M_Z)^2$, in terms of an integral in euclidian space with a cutoff at M_Z

$$\begin{aligned} X_6^r(\mu) - 4K_{12}^r(\mu) \simeq & \frac{1}{32\pi^2}\int_0^{M_Z^2} dx\,[\Gamma_{VV}(-x) + \Gamma_{AA}(-x)] \\ & + \frac{1}{16\pi^2}\left[-6\log\frac{M_Z}{\mu} + \frac{5}{2}\right] . \end{aligned} \quad (96)$$

In this form, it becomes possible to account for the logarithmic terms in the asymptotic behaviour of $\Gamma_{VV} + \Gamma_{AA}$ without encountering any divergence. We will make use of this feature in the next section.

At this point, we have discussed all the LEC's introduced in ref. [17] except for X_4 and X_7. Concerning the former, the corresponding term in the chiral Lagrangian involves only leptons and sources and bears no relevance for physical low-energy processes. The LEC X_7 has a decomposition given by eq. (55). In this expression, the LEC $\hat{X}_7$ parameterises the electromagnetic contribution to the lepton mass, which is not an observable quantity.

4 Minimal consistent resonance model

4.1 Estimation of the chiral couplings

The previous results can be applied to estimate numerically the chiral coupling constants which may be of physical relevance. One expects that major contributions in the integrands should come from light, narrow, resonances, which brings naturally to construct resonance models for the various form-factors. This idea was put into practice in the case of the form-factor Π_{VV-AA} a long time ago by Weinberg [29]. He showed that a minimal model comprising the π, ρ and a_1 resonances could yield reasonable results and satisfy the leading QCD asymptotic constraints (which determine all the resonance coupling constants in terms of the masses). In this model Π_{VV-AA} reads

$$\Pi_{VV-AA}(k^2) = \frac{M_A^2 M_V^2}{k^2(k^2 - M_V^2)(k^2 - M_A^2)} \ . \tag{97}$$

This resonance model was applied to the sum rule calculating the $\pi^+ - \pi^0$ mass difference [19] and gives a very accurate result. The generalisation of this minimal resonance model to the form factors F and G was discussed in ref. [18]

$$F(p^2, q^2) = \frac{p^2 - q^2 + 2(M_A^2 - M_V^2)}{2(p^2 - M_V^2)(q^2 - M_A^2)} \ , \quad G(p^2, q^2) = \frac{-q^2 + 2M_A^2}{(p^2 - M_V^2)(q^2 - M_A^2)q^2} \ , \tag{98}$$

and the form factors Γ_{VV} and Γ_{AA} were discussed in ref. [25]

$$\begin{aligned} \Gamma_{VV}(k^2, k.r) &= \frac{2k^2 - 2k.r - c_V}{2(k^2 - M_V^2)((r-k)^2 - M_V^2)} \ , \\ \Gamma_{AA}(k^2, k.r) &= \frac{2k^2 - 2k.r - c_A}{2(k^2 - M_A^2)((r-k)^2 - M_A^2)} \end{aligned} \tag{99}$$

(see also refs. [30, 31] for related work). The values of c_V and c_A are determined by the Wess-Zumino-Witten anomalous Lagrangian [32],

$$c_V = \frac{N_c M_V^4}{4\pi^2 F_0^2} \ , \qquad c_A = \frac{N_c M_A^4}{12\pi^2 F_0^2} \ . \tag{100}$$

Approximating correlators with rational functions is justified in the large-N_c limit [33]. But it is not really known whether only retaining the very first few poles should yield an accurate approximation of the actual Green functions. One can think of systematically improving on the minimal model by including more resonance poles together with more asymptotic constraints (see e.g. [25]). This interesting possibility is left for future work, and we stick to the minimal approximation in this paper.

Computing the integrals of sec. 3.5 in the minimal resonance model is straightforward. If we denote the ratio of the a_1 and ρ resonance masses $z = M_A^2/M_V^2$, we obtain for X_1, $\hat{X}_1$ and X_3

$$\begin{aligned} X_1 &= -\frac{3}{8}\frac{1}{16\pi^2}\left(\log(z) + \frac{c_V z - c_A}{2M_V^2 z}\right) , \\ \hat{X}_1 &= \frac{3}{8}\frac{1}{16\pi^2}\left(-2\log\frac{M_Z^2}{M_V^2} + \log(z) - \frac{c_V z + c_A}{2M_V^2 z} + 2\right) , \\ X_3^r(\mu) &= \frac{3}{2}\frac{1}{16\pi^2}\left(\log\frac{\mu^2}{M_V^2} + \frac{\log(z)}{z-1} - \frac{1}{6}\right) . \end{aligned} \tag{101}$$

For X_2^r and $\hat{X}_2^r$ one gets

$$\begin{aligned} X_2^r(\mu) &= \frac{3}{8}\frac{1}{16\pi^2}\left(\log\frac{\mu^2}{M_V^2} + \frac{\log(z)}{z-1} + \frac{5}{6}\right) , \\ \hat{X}_2^r(\mu) &= \frac{1}{8}\frac{1}{16\pi^2}\left(-10\log\frac{\mu_0^2}{M_V^2} + 7\log\frac{\mu^2}{M_V^2} + \frac{3(z+1)\log(z)}{(z-1)^2} - \frac{6z}{z-1} + \frac{5}{2}\right) \\ &\quad - \frac{1}{4}g_{02}^r(\mu_0) + \frac{1}{4}g_{03}^r(\mu_0) . \end{aligned} \tag{102}$$

Finally, the physical combination $X_6 - 4K_{12}$ reads

$$X_6^r(\mu) - 4K_{12}^r(\mu) = \frac{1}{16\pi^2}\left(-8\,\log\frac{M_Z}{M_V} + 3\log\frac{\mu^2}{M_V^2} + \frac{1}{2}\log(z) - \frac{c_V z + c_A}{4M_V^2 z} + \frac{7}{2}\right) . \tag{103}$$

This expression accounts for the contribution of the light resonances. In the asymptotic region ($k^2 > \mu_2^2$ with $\mu_2 \simeq 2$ GeV), however, our resonance model becomes inaccurate. In particular, while it reproduces (by construction) the leading asymptotic behaviour of $\Gamma_{VV} + \Gamma_{AA}$ it does not generate the logarithmic correction proportional to α_s (see eq.(95). We can estimate the modification in the value of X_6^{eff} induced by this effect following ref. [10] and the discussion in sec. 3.5.2. We content ourselves with an unsophisticated leading order expression for α_s. Then, from eqs. (95) (96) an analytical evaluation for this correction is obtained

$$\left(X_6^{eff}\right)_{\alpha_s} \simeq \frac{1}{4\pi^2\beta_0}\left[\log\left(\log\frac{M_Z^2}{\Lambda_{QCD}^2}\right) - \log\left(\log\frac{\mu_2^2}{\Lambda_{QCD}^2}\right)\right] . \tag{104}$$

In practice, we will use $\beta_0 = 11 - \frac{2}{3}N_f$ with $N_f = 4$ and $\Lambda_{QCD} = 206$ MeV which corresponds to $\alpha_s(m_\tau^2) = 0.35$ and $\alpha_s(M_Z^2) = 0.124$. The numerical value of $\left(X_6^{eff}\right)_{\alpha_s}$ is

$10^3\,X_1$	$10^3\,X_2^r$	$10^3\,X_3^r$	$10^3\,\tilde{X}_6^{eff}$	$10^3(X_6^{eff})_{\alpha_s}$	$10^3\,X_6^{eff}$
-3.7	3.6	5.00	10.4	3.0	-231.5

Table 1: *Numerical values of the physically relevant LEC's in the minimal resonance model with $\mu = M_V = 0.77$ GeV and $M_A^2/M_V^2 = 2$. In the case of X_6^{eff} we show separately the resonance contribution without the large logarithm (column 4), with the large logarithm (column 6) and in column 5 the perturbative α_s correction (see text).*

shown in table 1. The table also shows the numerical values of the LEC's generated by the minimal resonance model (with $z = 2$, $\mu = M_V = 0.77$ GeV). In the case of X_6^{eff} we observe that the α_s correction is sizable but the resonance contribution dominates. Both contributions have the same sign which is opposite to that of the large logarithm.

4.2 Examples of applications

Let us select a few applications of our results for illustrative purposes. To begin with, let us evaluate the Marciano-Sirlin constant C_1 which appears in the π_{l2} decay amplitude [34]. Comparing with the one-loop amplitude in chiral perturbation theory, Knecht *et al.* [17] have derived the decomposition of C_1 in terms of LEC's and chiral logarithms

$$\begin{aligned} C_1 = -4\pi^2 \Big[& \frac{8}{3}(K_1^r + K_2^r) + \frac{20}{9}(K_5^r + K_6^r) \\ & - \frac{4}{3}X_1 + 4(-X_2^r + X_3^r) - (\tilde{X}_6^r - 4K_{12}^r) \Big]_{\mu = M_\rho^2} \\ & + \frac{Z}{4}\left(3 + 2\log\frac{M_\pi^2}{M_\rho^2} + \log\frac{M_K^2}{M_\rho^2}\right) - \frac{1}{2} \end{aligned} \qquad (105)$$

where (following [7]) $\tilde{X}_6^r$ is defined as X_6^r minus the large logarithm. All the LEC's participating in this expression have been estimated on the basis of the consistent minimal resonance model. The $O(p^2)$ LEC Z is given in terms of $\langle VV - AA \rangle$ by the Das *et al.* sum rule [19], which yields in the minimal resonance model

$$Z = \frac{3}{2}\frac{1}{16\pi^2}\frac{M_V^2}{F_0^2}\frac{z\log(z)}{z-1} \simeq 0.92 \ . \qquad (106)$$

The sums $K_1 + K_2$ and $K_5 + K_6$ have been evaluated through the modeling of a set of QCD 4-point functions [37]. Combining these results with the estimates presented in this paper (table 1), we obtain

$$C_1 \simeq -0.93 - 1.63 = -2.56 \qquad (107)$$

where the first contribution comes from the LEC's K_i and X_i. In our opinion, the uncertainty on our estimates of these should not exceed 50%, which gives for the error on C_1

$$\Delta C_1 \simeq 0.5 \ . \qquad (108)$$

Our result for C_1 lies at the margin of the range guessed in ref. [34], $-2.4 \leq C_1 \leq 2.4$. It can be applied to extract a slightly more precise value of the pion decay constant F_π. Starting from eq. (21) of ref. [34],

$$\sqrt{2}F_\pi = 130.7 \left(\frac{0.9750}{V_{ud}}\right) \pm 0.1 + 0.15\, C_1 \text{ MeV} \tag{109}$$

and using an updated value for V_{ud} from ref. [35], we obtain

$$F_\pi = 92.2 \pm 0.2 \text{ MeV} . \tag{110}$$

As a second application, let us consider the ratio of the form factors arising in K^0_{l3} and K^+_{l3} decays. It was noted in ref. [1] that the only unknown input in one-loop chiral perturbation theory is the LEC X_1

$$\left.\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)}\right|_{ChPT} \equiv r^{th}_{0+} = 1.022 \pm 0.003 - 16\pi\alpha X_1 . \tag{111}$$

Our estimate for X_1 induces only very limited changes, giving $r^{th}_{0+} = 1.023 \pm 0.003$ which remains somewhat incompatible with the present experimental determination [2, 4, 5, 6] $r^{exp}_{0+} = 1.038 \pm 0.007$ (see [36]). Let us stress that the determination of X_1 should be reasonably accurate, since it involves the difference $VV - AA$ in an integral relation with a rapid convergence.

5 Conclusions

In this paper, we have studied the matching of the Standard Model to the chiral Lagrangian describing the dynamics of its lightest degrees of freedom at low energies. The high-energy dynamics of the Standard Model is encoded into the low-energy constants (LEC's) which are factors of local counterterms in the chiral Lagrangian. We have focused on the LEC's X_i that describe radiative corrections to weak semi-leptonic decays.

To determine the connection between these LEC's and the Standard Model, we have followed a two-step procedure. We started from the decay amplitude of a lepton, computed at one loop in the Standard Model, and we matched it onto the corresponding computation in Fermi theory. This has allowed us to determine the relevant counterterms in the latter theory. Then comes the second step of our matching procedure, from Fermi theory to the chiral effective Lagrangian. Thanks to a set of correlators defined within a spurion framework, we have related these Fermi counterterms to the chiral LEC's. This led us to generate for all the X_i's of physical relevance an integral representation involving two- and three-point Green functions of vector and axial-vector currents defined in chiral QCD. These can be brought into fairly simple forms involving just three form-factors: Γ_{VV}, Γ_{AA} and Π_{VV-AA}. Simple but non-trivial relationships among the chiral LEC's are revealed by these representations.

We dwelt on the case of X_6, whose representation involves a combination of Fermi counterterms left undetermined by the first step of our matching procedure. This indicated that this LEC should always appear in physical processes together with another chiral coupling, namely the electromagnetic LEC K_{12}, and we have derived an integral representation for the physical combination of the two LEC's. In practice this universal term is dominated by Sirlin's large logarithm. Our approach allows one to identify the unenhanced terms as well.

Finally, we have estimated the values of the X_i's by plugging into the integral representations a resonance model for the chiral two- and three-point Green functions. We have investigated the minimal resonance model that satisfies the leading asymptotic QCD constraints, with poles corresponding to Goldstone boson, vector and axial-vector resonances. Such a model is expected to yield rather accurate results whenever the sum rules are rapidly converging. In the case of the coupling X_6^{eff}, this criterion fails to be satisfied, and we have accounted for the main correction using perturbative QCD. Table 1 shows that the resonance contributions are smaller than the large logarithms by approximately a factor of twenty.

We presented two applications of our results. First, we reexamined the Marciano-Sirlin constant C_1, whose value lies slightly out of the range guessed in ref. [34]. A second outcome of our analysis concerns K_{l3} decays, for which various sets of data exist but are barely compatible within experimental errors. A good test of consistency consisted in the ratio of K_{l3}^0 and K_{l3}^+ form factors. Our estimate of X_1, based on a resonance model for the $VV - AA$ correlator, is too small to account for the discrepancy between experimental data and chiral perturbation theory.

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References

[1] V. Cirigliano, H. Neufeld and H. Pichl, Eur. Phys. J. C **35** (2004) 53 [hep-ph/0401173].

[2] A. Sher *et al.*, Phys. Rev. Lett. **91** (2003) 261802 [hep-ex/0305042].

[3] O. P. Yushchenko *et al.*, Phys. Lett. B **589** (2004) 111 [hep-ex/0404030].

[4] A. Lai *et al.* [NA48 Collaboration], Phys. Lett. B **602** (2004) 41 [hep-ex/0410059]; Phys. Lett. B **604** (2004) 1 [hep-ex/0410065].

[5] T. Alexopoulos *et al.* [KTeV Collaboration], Phys. Rev. Lett. **93** (2004) 181802 [hep-ex/0406001]; Phys. Rev. D **70** (2004) 092007 [hep-ex/0406003].

[6] P. Franzini, hep-ex/0408150.

[7] V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C **23** (2002) 121 [hep-ph/0110153].

[8] T. C. Andre, hep-ph/0406006,

V. Bytev, E. Kuraev, A. Baratt and J. Thompson, Eur. Phys. J. C **27** (2003) 57 [Erratum-ibid. C **34** (2004) 523] [hep-ph/0210049].

[9] S. M. Berman, Phys. Rev. Lett. **1** (1958) 468,

T. Kinoshita, Phys. Rev. Lett. **2** (1959) 477.

[10] A. Sirlin, Rev. Mod. Phys. **50** (1978) 573 [Erratum-ibid. **50** (1978) 905].

[11] A. Sirlin, Nucl. Phys. B **196** (1982) 83.

[12] S. Weinberg, Physica A **96** (1979) 327.

[13] J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142.

[14] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

[15] J. F. Donoghue, E. Golowich, B. H. Holstein, "Dynamics of the Standard Model", Cambridge University Press.

[16] R. Urech, Nucl. Phys. B **433** (1995) 234 [hep-ph/9405341].

[17] M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C **12** (2000) 469 [hep-ph/9909284].

[18] B. Moussallam, Nucl. Phys. B **504** (1997) 381 [hep-ph/9701400].

[19] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low and J. E. Young, Phys. Rev. Lett. **18** (1967) 759.

[20] J. Gasser, A. Rusetsky and I. Scimemi, Eur. Phys. J. C **32** (2003) 97 [hep-ph/0305260].

[21] E. Braaten and C. S. Li, Phys. Rev. D **42** (1990) 3888.

[22] C. Itzykson and J.B. Zuber, "Quantum field theory", McGraw-Hill (1980), sec. 5-1-3.

[23] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B **321** (1989) 311.

[24] M. J. Savage, M. E. Luke and M. B. Wise, Phys. Lett. B **291** (1992) 481 [hep-ph/9207233].

[25] M. Knecht and A. Nyffeler, Eur. Phys. J. C **21** (2001) 659 [hep-ph/0106034].

[26] S. Weinberg, "The quantum theory of fields", Cambridge University Press (1996), vol. 2, sec. 20.5.

[27] T. Das, V. S. Mathur and S. Okubo, Phys. Rev. Lett. **19** (1967) 859.

[28] V. Cuplov and A. Nehme, hep-ph/0311274.

[29] S. Weinberg, Phys. Rev. Lett. **18** (1967) 507.

[30] J. Bijnens, E. Gamiz, E. Lipartia and J. Prades, JHEP **0304** (2003) 055 [hep-ph/0304222].

[31] V. Cirigliano, G. Ecker, M. Eidemuller, A. Pich and J. Portoles, Phys. Lett. B **596** (2004) 96 [hep-ph/0404004].

[32] J. Wess and B. Zumino, Phys. Lett. B **37** (1971) 95, E. Witten, Nucl. Phys. B **223** (1983) 422.

[33] G. 't Hooft, Nucl. Phys. B **72** (1974) 461.

[34] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **71** (1993) 3629.

[35] A. Czarnecki, W. J. Marciano and A. Sirlin, Phys. Rev. D **70** (2004) 093006 [arXiv:hep-ph/0406324].

[36] H. Neufeld, Proceedings of the 337th WE-Heraeus seminar, Bad Honnef, Germany, December 13-17, 2004. , "Effective field theories in nuclear, particle and atomic physics" J. Bijnens, U. G. Meissner and A. Wirzba eds, hep-ph/0502008.

[37] B. Ananthanarayan and B. Moussallam, JHEP **0406** (2004) 047 [hep-ph/0405206].